

Hamiltonian Analysis of the Conformal Decomposition of the Gravitational Field

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ABSTRACT: This short note is devoted to the Hamiltonian formulation of the conformal decomposition of the gravitational field that was performed in [gr-qc/0501092]. We also analyze the gauge fixed form of the theory when we fix the conformal symmetry by imposing the condition $\sqrt{g} = 1$.

KEYWORDS: Hamiltonian Formalism, General Relativity.

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1. Introduction and Summary

The conformal-traceless decomposition of the gravitational field was firstly performed in [1] in its initial value problem ¹. The conformal traceless decomposition is defined by

$$h_{ab} = \phi^4 g_{ab} , \quad K_{ab} = \phi^{-2} A_{ab} + \frac{1}{3} g_{ab} \tau , \quad (1.1)$$

where h_{ab} is spatial physical metric and where K_{ab} is physical extrinsic curvature. We see that this definition is redundant since the multiple of the fields $g_{ab}, \phi, A_{ab}, \tau$ give the same physical metric g_{ab} and extrinsic curvature K_{ab} . In fact, we see that the decomposition (1.1) is invariant under the conformal transformation

$$\begin{aligned} g'_{ab}(\mathbf{x}, t) &= \Omega^4(\mathbf{x}, t) g_{ab}(\mathbf{x}, t) , & \phi'(\mathbf{x}, t) &= \Omega^{-1}(\mathbf{x}, t) \phi(\mathbf{x}, t) , \\ A'_{ab}(\mathbf{x}, t) &= \Omega^{-2}(\mathbf{x}, t) A_{ab}(\mathbf{x}, t) , & \tau'(\mathbf{x}, t) &= \tau(\mathbf{x}, t) , \end{aligned} \quad (1.2)$$

where $\mathbf{x} = (x^a, a = 1, 2, 3)$. We also see that (1.1) is invariant under following transformation

$$\tau'(\mathbf{x}, t) = \tau(\mathbf{x}) + \zeta(\mathbf{x}, t) , \quad A'_{ab}(\mathbf{x}, t) = A_{ab}(\mathbf{x}, t) - \frac{1}{3} \zeta(\mathbf{x}, t) \phi^6 g_{ab}(\mathbf{x}, t) . \quad (1.3)$$

Clearly the gauge fixing of these symmetries we can eliminate τ and ϕ . It is important to stress that in many recent applications the conformal invariance is broken by imposing the condition $\sqrt{g} = 1$. Further, the second symmetry (1.3) can be eliminated by imposing the traceless condition $A_{ab} g^{ab} = 0$. Then the variable τ is the trace of the extrinsic curvature $K = K_{ab} h^{ab}$. In what follows we do not impose neither from these conditions.

In this short note we perform the explicit Hamiltonian analysis of the conformal traceless decomposition of the gravitational field given in (1.1) following [4]. We start with the General Relativity action where the dynamical field is the physical metric h_{ab} . Then we find corresponding Hamiltonian and then express the action in the Hamiltonian form.

¹For review and extensive list of references, see [2].

As the next step we introduce the conformal traceless decomposition (1.1) into this action, identify the canonical variables and conjugate momenta and determine corresponding Hamiltonian and spatial diffeomorphism constraints. We also identify additional primary constraint that is a consequence of the symmetry of the theory under the transformation (1.2). As a result we find the Hamiltonian formulation of the General Relativity action that has an additional two degrees of freedom ϕ and its conjugate momenta p_ϕ together with an additional first class constraint.

As the next step we proceed to the analysis of the gauge fixing of the constraint that generates (1.2). Clearly imposing the gauge $\phi = 0$ we derive the standard General Relativity Hamiltonian. On the other hand when we introduce the gauge fixing function $\mathcal{G} : \sqrt{g} - 1 = 0$ we obtain theory where the physical degrees of freedom are the traceless components of the momenta together with the metric components that obey $\sqrt{g} = 1$. We also have the scalar and corresponding conjugate momenta that measures the scale factor of the metric. Finally the gauge fixing gives non-trivial Dirac brackets between these physical variables.

We mean that the gauge fixed form of the theory could be very useful for the formulation of some alternative versions of the theories of gravity. In particular we mean that it could be useful for the formulation of the Hořava-Lifshitz theory when we impose additional constraint on the scalar mode as in case of Non-Relativistic Covariant Hořava-Lifshitz gravity [7] or as in case of Lagrange multiplier modified Hořava-Lifshitz gravity [8]. It could be also useful for the formulation of the Spatially Covariant Theories of a Transverse, Traceless Graviton [9]. We hope that we proceed to the application of the formalism given in this paper for these specific problems in near future.

The structure of this note is as follows. In the next section (2) we find the Hamiltonian formalism for the conformal decomposition of gravity. We identify the primary and secondary constraints and calculate the Poisson brackets between them. Then in section (3) we perform the gauge fixing of the conformal symmetry and we find corresponding Hamiltonian and determine the Dirac brackets between phase space variables.

2. Hamiltonian Formalism for Conformal Decomposition of the Gravitational Field

Let us consider four dimensional action for General Relativity

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\hat{g}} {}^{(4)}R(\hat{g}) , \quad (2.1)$$

where ${}^{(4)}R$ is four dimensional scalar curvature evaluated on the four dimensional metric $\hat{g}_{\mu\nu}$. As the next step we perform 3 + 1 decomposition of the metric [3, 2]. Explicitly, we have following relation between four dimensional metric components $\hat{g}_{\mu\nu}$ and corresponding spatial metric components h_{ab} and the lapse N and shift functions N_a

$$\begin{aligned} \hat{g}_{00} &= -N^2 + N_a h^{ab} N_b , & \hat{g}_{0a} &= N_a , & \hat{g}_{ab} &= h_{ab} , \\ \hat{g}^{00} &= -\frac{1}{N^2} , & \hat{g}^{0a} &= \frac{N^a}{N^2} , & \hat{g}^{ab} &= h^{ab} - \frac{N^a N^b}{N^2} . \end{aligned} \quad (2.2)$$

Note also that 4-dimensional scalar curvature has following decomposition

$${}^{(4)}R = K_{ab}\mathcal{G}^{abcd}K_{cd} + R, \quad (2.3)$$

where R is three-dimensional spatial curvature, K_{ab} is extrinsic curvature defined as

$$K_{ab} = \frac{1}{2N}(\partial_t h_{ab} - \nabla_a N_b - \nabla_b N_a), \quad (2.4)$$

where ∇_a is covariant derivative built from the metric components h_{ab} . Finally we also ignored the boundary terms that are presented in (2.3).

Using this formalism we derive the General Relativity action in the form

$$S = \frac{1}{2\kappa^2} \int d^3\mathbf{x} dt \sqrt{h} N (K_{ab}\mathcal{G}^{abcd}K_{cd} + R), \quad (2.5)$$

where \mathcal{G}^{ijkl} is de Witt metric defined as

$$\mathcal{G}^{abcd} = \frac{1}{2}(h^{ac}h^{bd} + h^{ad}h^{bc}) - h^{ab}h^{cd} \quad (2.6)$$

with inverse

$$\mathcal{G}_{abcd} = \frac{1}{2}(h_{ac}h_{bd} + h_{ad}h_{bc}) - \frac{1}{2}h_{ab}h_{cd}, \quad \mathcal{G}_{abcd}\mathcal{G}^{cdmn} = \frac{1}{2}(\delta_a^m\delta_b^n + \delta_a^n\delta_b^m). \quad (2.7)$$

However for further purposes we introduce the generalized form of the de Witt metric that depends on the parameter λ as

$$\mathcal{G}^{abcd} = \frac{1}{2}(h^{ac}h^{bd} + h^{ad}h^{bc}) - \lambda h^{ab}h^{cd}, \quad \mathcal{G}_{abcd} = \frac{1}{2}(h_{ac}h_{bd} + h_{ad}h_{bc}) - \frac{\lambda}{3\lambda - 1}h_{ab}h_{cd}. \quad (2.8)$$

This generalized de Witt metric could be useful for the possible application of given procedure for more general theories of gravity as for example Hořava-Lifshitz gravity [5, 6].

In order to perform the Hamiltonian analysis of the conformal decomposition of the action (2.5) we firstly rewrite the action (2.5) its Hamiltonian form. To do this we introduce the conjugate momenta

$$P^{ab} = \frac{\delta S}{\delta \partial_t h_{ab}} = \frac{1}{2\kappa^2} \sqrt{h} \mathcal{G}^{abcd} K_{cd}, \quad P_N = \frac{\delta S}{\delta \partial_t N} = 0, \quad P_a = \frac{\delta S}{\delta \partial_t N^a} = 0. \quad (2.9)$$

Then we easily determine corresponding Hamiltonian

$$H = \int d^3\mathbf{x} (\partial_t h_{ab} P^{ab} - \mathcal{L}) = \int d^3\mathbf{x} (N \mathcal{H}'_T + N^a \mathcal{H}'_a), \quad (2.10)$$

where

$$\mathcal{H}'_T = \frac{2\kappa^2}{\sqrt{h}} P^{ab} \mathcal{G}_{abcd} P^{cd} - \frac{1}{2\kappa^2} \sqrt{h} R, \quad \mathcal{H}'_a = -2h_{ab} \nabla_c \pi^{cb}. \quad (2.11)$$

Using the Hamiltonian and the corresponding canonical variables we write the action (2.5) as

$$S = \int dt L = \int dt d^3\mathbf{x} (P^{ab} \partial_t h_{ab} - \mathcal{H}) = \int dt d^3\mathbf{x} (P^{ab} \partial_t h_{ab} - N \mathcal{H}'_T - N^a \mathcal{H}'_a). \quad (2.12)$$

Then we insert the decomposition (1.1) into the definition of the canonical momenta P^{ab}

$$P^{ab} = \frac{1}{2\kappa^2} \sqrt{g} (\phi^{-4} \tilde{\mathcal{G}}^{abcd} A_{cd} + \frac{1}{3} \phi^2 \tau \tilde{\mathcal{G}}^{abcd} g_{cd}) \quad (2.13)$$

where the metric $\tilde{\mathcal{G}}^{abcd}$ is defined as

$$\tilde{\mathcal{G}}^{abcd} = \frac{1}{2} (g^{ac} g^{bd} + g^{ad} g^{bc}) - \lambda g^{ab} g^{cd}, \quad \mathcal{G}^{abcd} = \phi^{-8} \tilde{\mathcal{G}}^{abcd}. \quad (2.14)$$

Note that $\tilde{\mathcal{G}}^{abcd}$ has the inverse

$$\tilde{\mathcal{G}}_{abcd} = \frac{1}{2} (g_{ac} g_{bd} + g_{ad} g_{bc}) - \frac{\lambda}{3\lambda - 1} g_{ab} g_{cd}, \quad \tilde{\mathcal{G}}_{abcd} = \phi^8 \mathcal{G}_{abcd}. \quad (2.15)$$

Using (2.13) and (1.1) we rewrite $P^{ab} \partial_t h_{ab}$ into the form

$$\begin{aligned} P^{ab} \partial_t h_{ab} &= \left(\frac{1}{2\kappa^2} \sqrt{g} \tilde{\mathcal{G}}^{abcd} A_{cd} + \frac{\sqrt{g}}{6\kappa^2} \phi^6 (1 - 3\lambda) \tau g^{ab} \right) \partial_t g_{ba} + \\ &+ \left(\frac{2}{\kappa^2} \sqrt{g} \phi^{-1} A_{ab} g^{ba} (1 - 3\lambda) + \frac{2\sqrt{g}}{\kappa^2} (1 - 3\lambda) \phi^5 \tau \right) \partial_t \phi. \end{aligned} \quad (2.16)$$

We see that it is natural to identify the expression in the parenthesis with momentum π^{ab} conjugate to g_{ab} and p_ϕ conjugate to ϕ respectively

$$\begin{aligned} \pi^{ab} &= \frac{1}{2\kappa^2} \sqrt{g} \tilde{\mathcal{G}}^{abcd} A_{cd} + \frac{\sqrt{g}}{6\kappa^2} (1 - 3\lambda) \phi^6 \tau g^{ab}, \\ p_\phi &= \frac{2}{\kappa^2} \sqrt{g} \phi^{-1} A_{ab} g^{ba} (1 - 3\lambda) + \frac{2\sqrt{g}}{\kappa^2} (1 - 3\lambda) \phi^5 \tau. \end{aligned} \quad (2.17)$$

Note that we do not impose the traceless condition $g_{ab} A^{ab} = 0$. However using (2.17) we can eliminate τ and determine following primary constraint

$$\Sigma_D : p_\phi \phi - 4\pi^{ab} g_{ba} = 0. \quad (2.18)$$

As we will see below this is the constraint that generates conformal transformation of the dynamical fields. Further, using (2.17) we find the relation between P^{ab} and π^{ab} in the form ²

$$P^{ab} = \phi^{-4} \pi^{ab}. \quad (2.21)$$

²It is important to stress that we could proceed exactly as in [4] and consider decomposition of the expression $P^{ab} \partial_t h_{ab}$ in the form

$$\begin{aligned} P^{ab} \partial_t h_{ab} &= \frac{2}{\kappa^2} \sqrt{g} \phi^{-1} A_{ab} g^{ba} (1 - 3\lambda) \partial_t \phi + \frac{1}{2\kappa^2} \sqrt{g} \tilde{\mathcal{G}}^{abcd} A_{ab} \partial_t g_{cd} + \\ &+ \frac{2\sqrt{g}}{\kappa^2} (1 - 3\lambda) \phi^5 \tau \partial_t \phi + \frac{1}{3\kappa^2} \phi^6 (1 - 3\lambda) \tau \partial_t \sqrt{g}. \end{aligned} \quad (2.19)$$

From (2.19) we see that it is natural to introduce an additional dynamical variable Q defined as $Q = \sqrt{g}$

Then we find that the kinetic term in the Hamiltonian constraint \mathcal{H}'_T takes the form

$$\frac{2\kappa^2}{\sqrt{h}}P^{ab}\mathcal{G}_{abcd}P^{cd} = \frac{2\kappa^2\phi^{-6}}{\sqrt{g}}\pi^{ab}\tilde{\mathcal{G}}_{abcd}\pi^{cd} . \quad (2.22)$$

As the next step we introduce the decomposition (1.1) into the contribution $\int d^3\mathbf{x}N^a\mathcal{H}_a$. Using the relation between Levi-Civita connections evaluated with the metric components h_{ab} and g_{ab}

$$\Gamma_{ab}^c(h) = \Gamma_{ab}^c(g) + 2\frac{1}{\phi}(\partial_a\phi\delta_b^c + \partial_b\phi\delta_a^c - \partial_d\phi g^{cd}g_{ab}) \quad (2.23)$$

and also if we define n_a through the relation $N_a = \phi^4 n_a$ we obtain

$$\int d^3\mathbf{x}N^a\mathcal{H}'_a = \int d^3\mathbf{x}n^a\mathcal{H}''_a , \quad (2.24)$$

where

$$\mathcal{H}''_a = -2g_{ad}D_b\pi^{bd} + 4\phi^{-1}\partial_a\phi g_{cd}\pi^{cd} , \quad (2.25)$$

where the covariant derivative D_a is defined using the Levi-Civita connection $\Gamma_{ab}^c(g)$. Observe that with the help of the constraint Σ_D we can write the constraint \mathcal{H}''_a as

$$\mathcal{H}''_a = -2g_{ac}D_b\pi^{bc} + \partial_b\phi p_\phi - 4\phi^{-1}\partial_a\phi\Sigma_D \equiv \hat{\mathcal{H}}_a - 4\phi^{-1}\partial_a\phi\Sigma_D \quad (2.26)$$

so that we see that it is natural to identify $\hat{\mathcal{H}}_a$ as an independent constraint. In fact, we will see that the smeared form of this constraint generates the spatial diffeomorphism.

Finally we proceed to the spatial curvature R . Note that there is a well known relation between $R[h]$ evaluated on h and $R[g]$ evaluated on g so that we find

$$-\frac{\sqrt{h}}{2\kappa^2}R[h] = -\frac{\sqrt{g}}{2\kappa^2}\phi^2R[g] + \frac{4\phi\sqrt{g}}{\kappa^2}g^{ab}D_aD_b\phi . \quad (2.27)$$

Collecting all these results together we obtain the Hamiltonian constraint in the form

$$\mathcal{H}'_T = \frac{2\kappa^2\phi^{-6}}{\sqrt{g}}\pi^{ab}\tilde{\mathcal{G}}_{abcd}\pi^{cd} - \frac{\sqrt{g}}{2\kappa^2}\phi^2R + \frac{4\sqrt{g}}{\kappa^2}\phi g^{ab}D_aD_b\phi \quad (2.28)$$

and identify corresponding conjugate momenta

$$\begin{aligned} p_\phi &= \frac{2}{\kappa^2}\sqrt{g}\phi^{-1}A_{ab}g^{ba}(1-3\lambda) + \frac{2\sqrt{g}}{\kappa^2}(1-3\lambda)\phi^5\tau , \\ \pi^{ab} &= \frac{1}{2\kappa^2}\sqrt{g}\tilde{\mathcal{G}}^{abcd}A_{cd} , \quad p_Q = \frac{1}{3\kappa^2}(1-3\lambda)\phi^6\tau . \end{aligned} \quad (2.20)$$

However when we introduce Q as an independent dynamical variable we should impose an additional primary constraint $Q - \sqrt{g} = 0$. On the other hand we mean that an existence of the additional primary constraint would make the analysis more complicated without any apparent advantages. For that reason we prefer the decomposition when we have dynamical variables g_{ab}, ϕ and corresponding conjugate momenta π^{ab}, p_ϕ respectively.

so that the action takes the form

$$S = \int dt d^3\mathbf{x} (\pi^{ab} \partial_t g_{ab} + p_\phi \partial_t \phi - n^a \hat{\mathcal{H}}_a - N \mathcal{H}'_T - \lambda \Sigma_D) , \quad (2.29)$$

where we included the primary constraint Σ_D multiplied by the Lagrange multiplier λ .

Now we can proceed to the Hamiltonian analysis of the conformal decomposition of the gravitational field given by the action (2.29). Clearly we have following primary constraints

$$\pi_N \approx 0 , \quad \pi_a \approx 0 , \quad \Sigma_D \approx 0 , \quad (2.30)$$

where π_N, π_a are momenta conjugate to N, n^a with following non-zero Poisson brackets

$$\{N(\mathbf{x}), \pi_N(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}) , \quad \{n^a(\mathbf{x}), \pi_b(\mathbf{y})\} = \delta_b^a \delta(\mathbf{x} - \mathbf{y}) . \quad (2.31)$$

Further, the preservation of the primary constraints π_N, π_a implies following secondary ones

$$\hat{\mathcal{H}}_a \approx 0 , \quad \mathcal{H}'_T \approx 0 . \quad (2.32)$$

Now we should analyze the requirement of the preservation of the primary constraint Σ_D during the time evolution of the system. First of all the explicit calculations give

$$\begin{aligned} \{\Sigma_D(\mathbf{x}), g_{ab}(\mathbf{y})\} &= 4g_{ab}(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) , \\ \{\Sigma_D(\mathbf{x}), \pi^{ab}(\mathbf{y})\} &= -4\pi^{ab}(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) , \\ \{\Sigma_D(\mathbf{x}), \phi(\mathbf{y})\} &= -\phi(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) , \\ \{\Sigma_D(\mathbf{x}), p_\phi(\mathbf{y})\} &= \phi(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (2.33)$$

using the canonical Poisson brackets

$$\{g_{ab}(\mathbf{x}), \pi^{cd}(\mathbf{y})\} = \frac{1}{2}(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c) \delta(\mathbf{x} - \mathbf{y}) , \quad \{\phi(\mathbf{x}), p_\phi(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}) . \quad (2.34)$$

It turns out that it is useful to introduce the smeared forms of the constraints $\mathcal{H}'_T, \hat{\mathcal{H}}_a, \Sigma_D$

$$\mathbf{T}_T(N) = \int d^3\mathbf{x} N \mathcal{H}'_T , \quad \mathbf{T}_S(N_a) = \int d^3\mathbf{x} N^a \hat{\mathcal{H}}_a , \quad \mathbf{D}(M) = \int d^3\mathbf{x} M \Sigma_D , \quad (2.35)$$

where N, N^a and M are smooth functions on \mathbf{R}^3 . Then using (2.33) and also

$$\{\Sigma_D(\mathbf{x}), \Gamma_{ab}^c(\mathbf{y})\} = 2\delta_b^c \partial_{y^a} \delta(\mathbf{x} - \mathbf{y}) + 2\delta_a^c \partial_{y^b} \delta(\mathbf{x} - \mathbf{y}) - 2g^{cd}(\mathbf{y}) \partial_{y^d} \delta(\mathbf{x} - \mathbf{y}) g_{ab}(\mathbf{y}) \quad (2.36)$$

we easily find that

$$\{\mathbf{D}(M), \mathcal{H}'_T(\mathbf{y})\} = 0 . \quad (2.37)$$

To proceed further we use following Poisson brackets

$$\begin{aligned}
\{\mathbf{T}_S(N^a), g_{ab}(\mathbf{x})\} &= -N^c \partial_c g_{ab}(\mathbf{x}) - \partial_a N^c g_{cb}(\mathbf{x}) - g_{ac} \partial_b N^c(\mathbf{x}) , \\
\{\mathbf{T}_S(N^a), \pi^{ab}(\mathbf{x})\} &= -\partial_c (N^c \pi^{ab})(\mathbf{x}) + \partial_c N^a \pi^{cb}(\mathbf{x}) + \pi^{ac} \partial_c N^b(\mathbf{x}) , \\
\{\mathbf{T}_S(N^a), \phi(\mathbf{x})\} &= -N^a \partial_a \phi(\mathbf{x}) , \\
\{\mathbf{T}_S(N^a), p_\phi(\mathbf{x})\} &= -\partial_a (N^a p_\phi)(\mathbf{x}) .
\end{aligned} \tag{2.38}$$

Then we easily find

$$\{\mathbf{T}_S(N^a), \Sigma_D(\mathbf{x})\} = -N^a \partial_a \Sigma_D(\mathbf{x}) - \partial_a N^a \Sigma_D(\mathbf{x}) \tag{2.39}$$

that together with (2.37) implies that $\Sigma_D \approx 0$ is the first class constraint.

Now we proceed to the analysis of the preservation of the secondary constraints $\mathcal{H}'_T \approx 0$, $\hat{\mathcal{H}}_a \approx 0$. In case of $\hat{\mathcal{H}}_a$ we find following Poisson brackets

$$\{\hat{\mathcal{H}}_a(\mathbf{x}), \hat{\mathcal{H}}_b(\mathbf{y})\} = \hat{\mathcal{H}}_b(\mathbf{x}) \frac{\partial}{\partial x^a} \delta(\mathbf{x} - \mathbf{y}) - \hat{\mathcal{H}}_a(\mathbf{y}) \frac{\partial}{\partial y^b} \delta(\mathbf{x} - \mathbf{y}) \tag{2.40}$$

which implies that the smeared form of the diffeomorphism constraints takes the familiar form

$$\{\mathbf{T}_S(N^a), \mathbf{T}_S(M^b)\} = \mathbf{T}_S(N^b \partial_b M^a - M^b \partial_b N^a) . \tag{2.41}$$

Further using (2.38) we easily find

$$\{\mathbf{T}_S(N^a), \mathcal{H}'_T(\mathbf{x})\} = -\partial_c N^c \mathcal{H}'_T(\mathbf{x}) - N^c \partial_c \mathcal{H}'_T(\mathbf{x}) \tag{2.42}$$

or equivalently

$$\{\mathbf{T}_S(N^a), \mathbf{T}_T(M)\} = \mathbf{T}_T(N^a \partial_a M) . \tag{2.43}$$

These results show that $\hat{\mathcal{H}}_a$ are the first class constraints.

Finally we have to calculate the Poisson brackets between Hamiltonian constraints. To do this we use the fact that

$$\{R(\mathbf{x}), \pi^{ab}(\mathbf{y})\} = -R^{ab}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) + D^a D^b \delta(\mathbf{x} - \mathbf{y}) - g^{ab} D_c D^c \delta(\mathbf{x} - \mathbf{y}) . \tag{2.44}$$

Then after some lengthy calculations we derive following Poisson bracket

$$\begin{aligned}
\{\mathbf{T}_T(N), \mathbf{T}_T(M)\} &= -\frac{8}{3\lambda - 1} \int d^3 \mathbf{x} (\partial_h N M - \partial_h M N) \phi^{-5} g^{hp} \partial_p \phi g_{cd} \pi^{cd} + \\
&+ 2 \int d^3 \mathbf{x} (\partial_d N M - \partial_d M N) \phi^{-4} D_c \pi^{cd} + \\
&+ 2 \frac{\lambda - 1}{3\lambda - 1} \int d^3 \mathbf{x} (\partial_h N M - \partial_h M N) g^{hm} D_m (g_{cd} \pi^{cd}) .
\end{aligned} \tag{2.45}$$

First of all we see that in order to eliminate the last term we have to demand that the parameter λ is equal to one. In what follows we will then presume that $\lambda = 1$ keeping

in mind that the general case of $\lambda \neq 1$ could be useful when we perform the conformal decomposition of the metric in case of Hořava-Lifshitz gravity. For $\lambda = 1$ we obtain that (2.45) can be written as

$$\begin{aligned} \{\mathbf{T}_T(N), \mathbf{T}_T(M)\} &= \\ &= \mathbf{T}_S((\partial_b MN - \partial_b NM)g^{ba}\phi^{-4}) + \mathbf{D}((\partial_a NM - \partial_a MN)\phi^{-5}g^{ab}\partial_b\phi) . \end{aligned} \quad (2.46)$$

This result implies that the Poisson bracket between Hamiltonian constraints vanishes on the constraint surface.

Now we can outline our results. We performed the Hamiltonian analysis of the action (2.29) and we identified the first class constraints $\pi_N \approx 0$, $\pi_a \approx 0$, $\mathcal{H}'_T \approx 0$, $\hat{\mathcal{H}}_a \approx 0$ and $\Sigma_D \approx 0$. On the other hand we have following phase space degrees of freedom $N, \pi_N, n^a, \pi_a, g_{ab}, \pi^{cd}$ and ϕ, p_ϕ . Then using the standard counting of the physical degrees of freedom [10] we find that given theory has four physical degrees of freedom which is the correct number of the degrees of freedom of the General Relativity.

3. Fixing Gauge Symmetry $\Sigma_D \approx 0$

We saw in previous section that conformal decomposition of the gravitational field implies an existence of the additional scalar field ϕ together with the first class constraint $\Sigma_D \approx 0$ that generates the conformal transformation. The simplest way how to fix given symmetry is to impose the constraint $\phi = 0$ which however leads to the standard General Relativity Hamiltonian. Clearly this is not very interesting result. For that reason we rather consider following form of the gauge fixing function

$$\mathcal{G}(\mathbf{x}) : \sqrt{g}(\mathbf{x}) - 1 = 0 . \quad (3.1)$$

In this case the scalar ϕ has the physical meaning as the scale factor of the metric. Now we would like to see the consequence of the gauge fixing (3.1) for the structure of the theory.

As the first step we should note that the extended Hamiltonian now contains the constraint \mathcal{G} which, in order to fix the gauge has to have non-zero Poisson brackets with Σ_D and also \mathcal{G} has to be preserved during the time evolution of the system. In fact, we have to check that all constraints are now preserved when the extended Hamiltonian contains the additional constraint $\mathcal{G} \approx 0$. Explicitly

$$H_T = \int d^3\mathbf{x} \left(N\mathcal{H}'_T + n^a\hat{\mathcal{H}}_a + v^N\pi_N + v^a\pi_a + \lambda\Sigma_D + \Gamma\mathcal{G} \right) . \quad (3.2)$$

Then using following Poisson bracket

$$\begin{aligned} \left\{ \sqrt{g}(\mathbf{x}), \pi^{ab}(\mathbf{y}) \right\} &= \frac{1}{2}g^{ab}\sqrt{g}(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) \\ \left\{ \mathcal{H}'_T(\mathbf{y}), \mathcal{G}(\mathbf{x}) \right\} &= \kappa^2\phi^{-6}g_{cd}\pi^{cd}(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) , \\ \left\{ \Sigma_D(\mathbf{y}), \mathcal{G}(\mathbf{x}) \right\} &= 6\sqrt{g}(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) \approx 6\delta(\mathbf{x} - \mathbf{y}) \equiv \Delta_{\Sigma_D, \mathcal{G}}(\mathbf{x}, \mathbf{y}) \end{aligned} \quad (3.3)$$

we find equations that determine the time evolution of the constraints

$$\begin{aligned}
\partial_t \mathcal{G}(\mathbf{x}) &= -\{H_T, \mathcal{G}(\mathbf{x})\} = \\
&= -\left(N\kappa^2\phi^{-6}g_{ab}\pi^{ba}(\mathbf{x}) + 6\sqrt{g}\lambda(\mathbf{x}) - \partial_a N^a(\mathbf{x}) - \partial_a N^a \mathcal{G}(\mathbf{x}) - N^a \partial_a \mathcal{G}(\mathbf{x})\right), \\
\partial_t \mathcal{H}'_T(\mathbf{x}) &= -\{H_T, \mathcal{H}'_T(\mathbf{x})\} \approx -\int d^3\mathbf{y} \Gamma(\mathbf{y}) \{\mathcal{G}(\mathbf{y}), \mathcal{H}'_T(\mathbf{x})\} = \Gamma\kappa^2\phi^{-6}g_{cd}\pi^{cd}(\mathbf{x}) = 0, \\
\partial_t \Sigma_D(\mathbf{x}) &= -\{H_T, \Sigma_D(\mathbf{x})\} \approx -\int d^3\mathbf{y} \Gamma(\mathbf{y}) \{\mathcal{G}(\mathbf{y}), \Sigma_D(\mathbf{x})\} = 6\sqrt{g}\Gamma(\mathbf{x}) = 0.
\end{aligned} \tag{3.4}$$

We see that the last two equations has the solution $\Gamma = 0$ while the first one gives

$$\lambda = -\frac{N\kappa^2\phi^{-6}g_{ab}\pi^{ba}}{6\sqrt{g}} + \frac{1}{6\sqrt{g}}\partial_a N^a. \tag{3.5}$$

Inserting (3.5) together with $\Gamma = 0$ into the extended Hamiltonian H_T we find

$$\begin{aligned}
H_T &= \int d^3\mathbf{x} \left(N \left[\mathcal{H}_T - \frac{\kappa^2\phi^{-6}}{6\sqrt{g}}g_{cd}\pi^{cd}\Sigma_D \right] + v^N\pi_N + v^a\pi_a \right) + \\
&+ \int d^3\mathbf{x} N^a \left(-2g_{ac}\nabla_d\pi^{cd} - \partial_a \left[\frac{1}{6\sqrt{g}}\Sigma_D \right] + p_\phi\partial_a\phi \right) \equiv \\
&\equiv \int d^3\mathbf{x} (N\tilde{\mathcal{H}}_T + N^a\tilde{\mathcal{H}}_a + v^N\pi_N + v^a\pi_a).
\end{aligned} \tag{3.6}$$

We claim that the Hamiltonian on the reduced phase space is given as the linear combinations of the first class constraints $\tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_a$ together with the second class constraints Σ_D, \mathcal{G} .

To see this explicitly we again introduce the smeared form of these constraints

$$\begin{aligned}
\tilde{\mathbf{T}}_T(N) &= \int d^3\mathbf{x} \tilde{\mathcal{H}}_T = \mathbf{T}_T(N) - \mathbf{D} \left(N \frac{\kappa^2\phi^{-6}}{6\sqrt{g}}g_{cd}\pi^{cd} \right), \\
\tilde{\mathbf{T}}_S(N^a) &= \int d^3\mathbf{x} N^a \tilde{\mathcal{H}}_a = \mathbf{T}_S(N^a) + \mathbf{D} \left(\frac{1}{6\sqrt{g}}\partial_a N^a \right).
\end{aligned} \tag{3.7}$$

Now using the Poisson brackets determined in previous section we see that the Poisson brackets between $\tilde{\mathbf{T}}_T, \tilde{\mathbf{T}}_S$ are proportional to the constraints and hence vanish on the constraint surface. It is also clear that we have

$$\left\{ \tilde{\mathbf{T}}_T(N), \mathcal{G}(\mathbf{x}) \right\} = 0, \quad \left\{ \tilde{\mathbf{T}}_S(N^a), \mathcal{G}(\mathbf{x}) \right\} = 0 \tag{3.8}$$

together with $\left\{ \tilde{\mathbf{T}}_T(N), \mathbf{D}(M) \right\} = 0, \left\{ \tilde{\mathbf{T}}_S(N^a), \mathbf{D}(M) \right\} = 0$ which show that $\tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_a$ are the first class constraints.

As the next step we have to eliminate the second class constraints which can be done when we replace the Poisson brackets with corresponding Dirac brackets. Explicitly we

find

$$\begin{aligned}
\left\{ g_{ab}(\mathbf{x}), \pi^{cd}(\mathbf{y}) \right\}_D &= \left\{ g_{ab}(\mathbf{x}), \pi^{cd}(\mathbf{y}) \right\} - \int d\mathbf{z} d\mathbf{z}' \left\{ g_{ab}(\mathbf{x}), \Sigma_D(\mathbf{z}) \right\} \Delta^{\Sigma_D, \mathcal{G}}(\mathbf{z}, \mathbf{z}') \left\{ \mathcal{G}(\mathbf{z}'), \pi^{cd}(\mathbf{y}) \right\} - \\
&\quad - \int d\mathbf{z} d\mathbf{z}' \left\{ g_{ab}(\mathbf{x}), \mathcal{G}(\mathbf{z}) \right\} \Delta^{\mathcal{G}, \Sigma_D}(\mathbf{z}, \mathbf{z}') \left\{ \Sigma_D(\mathbf{z}'), \pi^{cd}(\mathbf{y}) \right\} = \\
&= \frac{1}{2}(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c) \delta(\mathbf{x} - \mathbf{y}) - \frac{1}{3} g_{ab} g^{cd}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) ,
\end{aligned} \tag{3.9}$$

where $\Delta^{\mathcal{G}, \Sigma_D}$ is the inverse to $\Delta_{\Sigma_D, \mathcal{G}}$ with following non-zero components

$$\Delta^{\mathcal{G}, \Sigma_D}(\mathbf{x}, \mathbf{y}) = \frac{1}{6} \delta(\mathbf{x} - \mathbf{y}) , \quad \Delta^{\Sigma_D, \mathcal{G}}(\mathbf{x}, \mathbf{y}) = -\frac{1}{6} \delta(\mathbf{x} - \mathbf{y}) . \tag{3.10}$$

In the same way we find

$$\begin{aligned}
\left\{ \pi^{ab}(\mathbf{x}), \pi^{cd}(\mathbf{y}) \right\}_D &= \frac{1}{3} \left(\pi^{ab} g^{cd} - g^{ab} \pi^{cd} \right) (\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) , \\
\left\{ \pi^{ab}(\mathbf{x}), p_\phi(\mathbf{y}) \right\}_D &= \frac{1}{6} g^{ab} p_\phi(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) , \quad \left\{ \pi^{ab}(\mathbf{x}), p_\phi(\mathbf{y}) \right\}_D = -\frac{1}{6} g^{ab} p_\phi(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) .
\end{aligned} \tag{3.11}$$

If we define $\pi = \pi^{ab} g_{ba}$ we find

$$\left\{ g_{ab}(\mathbf{x}), \pi(\mathbf{y}) \right\}_D = 0 , \quad \left\{ \pi^{ab}(\mathbf{x}), \pi(\mathbf{y}) \right\}_D = 0 . \tag{3.12}$$

It turns out that it is useful to introduce the traceless part of the conjugate momentum $\tilde{\pi}^{ab}$

$$\tilde{\pi}^{ab} = \pi^{ab} - \frac{1}{3} g^{ab} \pi , \quad \tilde{\pi}^{ab} g_{ba} = 0 . \tag{3.13}$$

Using previous results we derive following Dirac brackets

$$\begin{aligned}
\left\{ g_{ab}(\mathbf{x}), \tilde{\pi}^{cd}(\mathbf{y}) \right\}_D &= \frac{1}{2}(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c) \delta(\mathbf{x} - \mathbf{y}) - \frac{1}{3} g_{ab} g^{cd}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) , \\
\left\{ \tilde{\pi}^{ab}(\mathbf{x}), \tilde{\pi}^{cd}(\mathbf{y}) \right\}_D &= \frac{1}{3} \left(\tilde{\pi}^{ab} g^{cd} - g^{ab} \tilde{\pi}^{cd} \right) (\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) , \\
\left\{ \tilde{\pi}^{ab}(\mathbf{x}), p_\phi(\mathbf{y}) \right\}_D &= 0 , \quad \left\{ \tilde{\pi}^{ab}(\mathbf{x}), \phi(\mathbf{y}) \right\}_D = 0 .
\end{aligned} \tag{3.14}$$

We showed that $\tilde{\mathcal{H}}_T$ together with $\tilde{\mathcal{H}}_a$ are the first class constraints. We also identified the second class constraints Σ_D, \mathcal{G} . According to the standard analysis of the constraint systems these constraints can be solved explicitly. Solving the constraint $\Sigma_D = 0$ for π we obtain $\pi_{ab} g^{ab} = \frac{1}{4} p_\phi \phi$ while solving the constraint \mathcal{G} gives $\sqrt{g} = 1$. Then the Hamiltonian constraint $\tilde{\mathcal{H}}_T$ takes the form

$$\tilde{\mathcal{H}}_T = 2\kappa^2 \phi^{-6} \tilde{\pi}^{ab} g_{ac} g_{bd} \tilde{\pi}^{cd} - \frac{1}{2\kappa^2} \phi^2 R + \frac{4}{\kappa^2} \phi g^{ab} D_a D_b \phi - \frac{\kappa^2}{24} \phi^{-4} p_\phi^2 . \tag{3.15}$$

Observe that $\tilde{\mathcal{H}}_T$ depends on $\tilde{\pi}^{ab}, g_{ab}$ that have 8 phase space degrees of freedom together with p_ϕ, ϕ . In the same way we find that $\tilde{\mathcal{H}}_a$ is equal to

$$\tilde{\mathcal{H}}_a = -2g_{ac}\nabla_a\tilde{\pi}^{cd} - \frac{1}{6}\partial_a(p_\phi\phi) + p_\phi\partial_a\phi. \quad (3.16)$$

It is interesting to determine the Dirac bracket between the smeared form of the constraints $\tilde{\mathcal{H}}_a$ and the canonical variables. In fact, since $\tilde{\mathcal{H}}_a$ are the first class constraints we find that the Dirac brackets between them and any phase space variable coincides with corresponding Poisson bracket. Then we obtain

$$\begin{aligned} \left\{ \tilde{\mathbf{T}}_S(N^m), g_{ab}(\mathbf{x}) \right\}_D &= -N^c\partial_c g_{ab}(\mathbf{x}) - \partial_a N^c g_{cb}(\mathbf{x}) - g_{ac}\partial_b N^c(\mathbf{x}) - \frac{2}{3}\partial_c N^c g_{ab}(\mathbf{x}), \\ \left\{ \tilde{\mathbf{T}}_S(N^m), \tilde{\pi}^{ab}(\mathbf{x}) \right\}_D &= -N^c\partial_c \tilde{\pi}^{ab}(\mathbf{x}) + \partial_c N^a \tilde{\pi}^{cb}(\mathbf{x}) + \tilde{\pi}^{ac}\partial_c N^b(\mathbf{x}) - \frac{1}{3}\partial_c N^c \tilde{\pi}^{ab}(\mathbf{x}), \\ \left\{ \tilde{\mathbf{T}}_S(N^m), \phi(\mathbf{x}) \right\}_D &= -N^c\partial_c \phi(\mathbf{x}) - \frac{1}{6}\partial_m N^m \phi(\mathbf{x}), \\ \left\{ \tilde{\mathbf{T}}_S(N^m), p_\phi(\mathbf{x}) \right\}_D &= -N^c\partial_c p_\phi(\mathbf{x}) - \frac{5}{6}\partial_c N^c p_\phi(\mathbf{x}). \end{aligned} \quad (3.17)$$

Then after some calculations we find

$$\left\{ \tilde{\mathbf{T}}_S(N^a), \tilde{\mathcal{H}}_T(\mathbf{x}) \right\} = -N^m\partial_m \tilde{\mathcal{H}}_T(\mathbf{x}) - \partial_m N^m \tilde{\mathcal{H}}_T(\mathbf{x}) \quad (3.18)$$

which is desired result since it shows that the Hamiltonian constraint transforms as the tensor density under spatial diffeomorphism generated by $\tilde{\mathbf{T}}_S(N^a)$.

Let us outline results derived in this section. We fix of the conformal symmetry by imposing the condition $\sqrt{g} = 1$. Then we find that the dynamical fields ϕ, p_ϕ together with $\tilde{\pi}^{ab}, g_{ab}$ where $\sqrt{g} = 1$ and where $\tilde{\pi}^{ab}g_{ba} = 0$. We also showed that given there are four first class constraints $\tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_a$. Then we can proceed further and perform the gauge fixing of some of these first class constraints. In fact, we can fix the Hamiltonian constraint $\tilde{\mathcal{H}}_T$ in order to eliminate the scalar field degrees of freedom p_ϕ, ϕ so that the reduced phase space will be governed by $g_{ab}, \tilde{\pi}^{ab}$ with the three first class constraints $\tilde{\mathcal{H}}_a$. Note that due to the presence of these constraints the number of physical degrees of freedom is four. We mean that the theory formulated with $g_{ab}, \tilde{\pi}^{ab}$ which is invariant under the spatial diffeomorphism could be the starting point for the alternative formulations of theory of gravity, see for example [9].

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